Summary of Generating Functions

Note: I will be abbreviating Generating Function as GF.

1. General Strategy / Method:

Step 1: Introduce a new var, x. There are 3 meanings for x:

a) We select 1 of the objects.
b) Add 1 to the value of the vars.
c) Perform one action. (Translating from combinatorics to algebra.)

Note: Performing multiple actions at the same time corresponds to "and" and "multiplication".

x^k How many times the action is performed.

Performing an action.

Step 2: List all our objects/vars that are under consideration, and for each of them, we derive the associated power series.

Step 3: To get all possibilities for our problem, we multiply all the power series we got in step 2.
Note: The product of power series is a power series. Hence, when we multiply all the power series, we get one big power series that produces all possibilities.

Step 4: Compute the coefficient of x^n in the power series we derived in step 3. The coefficient is the number of ways we can perform an action.

Consider $a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$.
This is the GF of a_n. We first compute the GF and then the coefficient.

2. Examples:

E.g. 1 How many solns $(x_1, x_2, \dots, x_n)$ are there for the following eqn:
$x_1 + x_2 + \dots + x_n = k$ where $x_i \in \{0,1\}$ for all $i = 1, 2, \dots, n$?

Soln:
1. Introduce a new var, x. The meaning of x is that we are adding 1 to the value of the x_i's.

2.

$x_1: x^0 + x \to 1+x$

$x_2: x^0 + x \to 1+x$

$\vdots$

$x_n: x^0 + x \to 1+x$

3.

Final power series: $(1+x)(1+x)\ldots(1+x)$
$= (1+x)^n$

4.

The answer is the coefficient of x^k.
However, we know that:

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$

Hence, the soln is $\binom{n}{k}$.

E.g. 2 How many solns $(x_1, \ldots, x_n)$ are there for the following eqn:
$x_1 + \ldots + x_n = k$ for all $x_i \in \mathbb{N}$, $x_i \geq 0$ for all $i = 1, 2, \ldots n$?

Soln:

1. We introduce a new var, x. The meaning of x is that we add 1 to one of the x_i's.

2.

$x_1 : 1 + x + x^2 + \ldots$

$x_2 : 1 + x + x^2 + \ldots$

$x_3 : 1 + x + x^2 + \ldots$

$\vdots$

$x_n : 1 + x + x^2 + \ldots$

3.

Combined PS: $(1 + x + \ldots + x^k + x^{k+1} + \ldots)^n$

4.

Answer: The coefficient of x^k.
We know that $x^{k+1} + x^{k+2} + \ldots$ will never be used. The "good" terms are $1 + x + x^2 + \ldots + x^k$.

Note: Include all terms in your ps, even ones that are unattainable.

$$1 + x + x^2 + \ldots + x^k = \sum_{k=0}^{\infty} \left[{n \atop k} \right] \cdot x^k$$ A thm

Hence, the soln is $\left[{n \atop k} \right]$.

E.g. 3 How many solns (x_1, x_2, x_3, x_4) are there to the eqn:
$x_1 + x_2 + 2x_3 + 3x_4 = n$ s.t.
$x_1 \geq 2$, x_2 is always even, $x_3 \geq 0$, $x_4 \geq 3$?

Soln:

1. Introduce a new var, x. The meaning of x is that it adds 1 to one of the X_i's.

2.

$X_1: x^2+x^3+\dots$

$X_2: 1+x^2+x^4+\dots$

$2X_3: 1+x^2+x^4+\dots$

$3X_4: x^9+x^{12}+x^{15}+\dots$

Note: We know that $X_3 \geq 0$, so $2X_3$ is also ≥ 0. Furthermore, $2X_3$ is an even number.

Note: $X_4 = 3, 4, 5, 6, \dots$ Hence, $3X_4 = 9, 12, 15, \dots$ (Multiples of $3 \geq 9$).

3.

Final PS: $(1+x+x^2+\dots)\cdot x^2\cdot(1+x^2+x^4+\dots)^2\cdot x^9\cdot(1+x^3+x^6+\dots)$

$$= x^{11}\underbrace{(1+x+x^2+\dots)(1+x^2+x^4+\dots)^2(1+x^3+x^6+\dots)}_{F(x)}$$

4.

$$F(x) = \sum_{n\geq 0} a_n\cdot x^n$$

E.g. 4 In how many ways can we toss a die 3 times s.t. the total is 14?

Soln:

1. We introduce a new var, x. The meaning of x is adding 1 to the total.

2. The vars are the results of each toss.

result (1) $= x + x^2 + \dots + x^6$

result (2) $= x + x^2 + \dots + x^6$

result (3) $= x + x^2 + \dots + x^6$

3.

Combined PS: $(x + \dots + x^6)^3$

We want the coefficient of x^{14}.

4.

To get the coefficient of x^{14}, we have to expand $(x + \dots + x^6)^3$.

E.g. 5 Find the num of ways we can select $2n$ balls from n identical red balls, n identical blue balls and n identical white balls. (There are $3n$ balls in total.)

Soln:

We will convert this problem to linear eqns.
Let X_1 = # of red balls we select.
Let X_2 = # of blue balls we select.
Let X_3 = # of white balls we select.

$X_1 + X_2 + X_3 = 2n$

$0 \le X_1, X_2, X_3 \le n$ ← Have at most n balls of each color.

1. We will introduce a new var, x. The meaning of x is that we add 1 to one of the vars.

2.

$$\left.\begin{aligned} X_1 &= 1 + x + x^2 + \dots + x^n \\ X_2 &= 1 + x + x^2 + \dots + x^n \\ X_3 &= 1 + x + x^2 + \dots + x^n \end{aligned}\right\} \rightarrow \frac{1-x^{n+1}}{1-x}$$

3.
Combined PS: $\left(\frac{1-x^{n+1}}{1-x}\right)^3$

4.

$$\left(\frac{1-x^{n+1}}{1-x}\right)^3 = \sum_{k \ge 0} \underbrace{a_k}_{\text{\# of ways to get } x_1 + x_2 + x_3 = k} \cdot x^k$$

Ans: a_{2n}

$$\sum_{k\geq 0} a_k \cdot x^k = \frac{(1-x^{n+1})^3}{(1-x)^3}$$

$$(1-x^{n+1})^3 = 1 - 3(x^{n+1}) + 3(x^{n+1})^2 - (x^{n+1})^3$$

$$\frac{1}{(1-x)} = 1 + x + x^2 + \cdots + x^n + \cdots$$

$$\frac{1}{(1-x)^2} = 1 + 2x + \cdots + n \cdot x^{n-1} \quad \Big\} \text{ Derivative of } \frac{1}{1-x}$$

$$\frac{1}{(1-x)^3} = 1 + 3x + \cdots + \frac{(n+1)(n+2)}{2} \cdot x^n \quad \Big\} \text{ Derivative of } \left(\frac{1}{1-x}\right)^2$$

$$\left(\frac{1-x^{n+1}}{1-x}\right)^3 = \left(1 - 3(x^{n+1}) + \cancel{3(x^{n+1})^2} - \cancel{(x^{n+1})^3}\right) \cdot \left(1 + 3x + \cdots + \frac{(n+1)(n+2)}{2} \cdot x^n\right)$$

There are 2 ways we can get x^{2n}:

1. $(1)\left(\frac{(2n+1)(2n+2)}{2} \cdot x^{2n}\right)$, in which case the coefficient is $\frac{(2n+1)(2n+2)}{2}$.

2. $(-3x^{n+1})\left(\frac{(n)(n+1)}{2}\right)x^{n-1}$, in which case the coefficient is $\left(\frac{-3}{2}\right)(n)(n+1)$.

Note: $\left.\begin{array}{l} 3(x^{n+1})^2 \Rightarrow 3(x^{2n+2}) \\ (-1)(x^{n+1})^3 \Rightarrow -(x^{3n+3}) \end{array}\right\} > x^{2n}$, so we can't use them.

$$\frac{(2n+1)(2n+2)}{2} + \frac{(-3)(n)(n+1)}{2}$$

$$= \frac{(n+1)(n+2)}{2}$$

$$= \underbrace{\binom{n+2}{2}}$$

Answer (Coefficient of x^{2n}).